DEVELOPING MATHEMATICAL CONCEPTS in VISUALLY HANDICAPPED PUPILS in the Secondary School

With an Introduction by
DR. ABRAHAM NEMETH

CINCINNATI PUBLIC SCHOOLS

230 East Ninth Street
Cincinnati, Ohio 45202

Developed as an E.S.E.A. Title VI Project
Under the Direction of the Division of Special Education

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PREPARED BY

GLORIA JACKSON
Aiken High School

MARY PINKELTON
Merry Junior High School

BOB R. HENSLEY
Project Coordinator

MURIEL C. LASS
Assistant Project Coordinator

CONSULTANTS

Mrs. Ferdinand J. Mann, Member of the Braille Authority

Roy D. Matthews, Supervisor, Secondary Mathematics, Cincinnati Public Schools

Dr. Abraham Nemeth, Associate Professor of Mathematics, Detroit University

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INTRODUCTION

In the past two decades, there has been a marked increase in the number of blind pupils who attend school in their own communities. Accordingly, resource and itinerant teachers, as well as regular classroom teachers, have been brought into closer and more frequent contact with the problems that blind children face. While the ingenuity of these teachers has been challenged to solve such problems in all areas, it has been especially taxed in the area of communicating junior and senior high school mathematics to blind pupils.

It is no wonder that this should be so, when we consider the fact that, on the teaching end, visual appeal, and on the learning end, manipulative skill, are more strongly emphasized than in other subjects. For some time now, there has been a clamor for some guidelines and procedures that might be of help in this situation. This manual takes a step in filling this need.

In consulting with the authors of this manual, I have taken the liberty of injecting my philosophy concerning the teaching of mathematics to blind children. In summary, that philosophy is as follows:

Admittedly, blind children, in general, grasp mathematical concepts later than their schoolmates who can see. But this is not because blind children are intellectually impaired; it is rather because contact with their environment has not been as rich as for normally seeing children of the same age, with an attendant delay in the formation of number concepts. With this in mind, the teacher must never take for granted that, because a blind child has attained a certain age, he must, as a matter of course, have mastered a particular concept or skill.

In the presentation of mathematics through the sense of touch, it must always be remembered that the mode of presentation is a model and an imitation of the corresponding situation as perceived through the sense of sight. If the blind child has no concept of the original, the model cannot have much meaning for him. It is therefore necessary, at every step, for the blind child to be made aware of accepted practices in the realm of vision; only then can the model presented to him through the sense of touch become meaningful. This is not to imply that a blind child should develop a skill in handwriting or drawing; it implies only that he be aware of what the results would be if he had these skills. These points will be emphasized over and over again as the various topics come up for discussion throughout the manual, and their precise application will be made clear.

It is my earnest wish that this manual be of value to those who come in contact with blind children in the area of mathematics. Teachers will then have a sense of achievement rather than of frustration, but, more importantly, the blind child will be helped to attain his potential and to have a rewarding educational experience.

Abraham Nemeth, Ph.D.

DEVELOPING MATHEMATICAL CONCEPTS IN VISUALLY HANDICAPPED PUPILS

Without a conscientious effort by both parents and teachers, visually handicapped pupils are apt to lack many experiences which build mathematical concepts. Sighted children build many of these concepts in play and as part of everyday living. They count telephone poles on trips, count blocks in the sidewalk, divide candy bars, compare sizes and shapes of people and objects, judge distance and velocity, and recognize that some sets contain more elements than others.

The visually handicapped pupil is often protected from many of these experiences and simply unaware of others. He may be carefully guided around objects that contribute to size and shape discrimination. Unless he is allowed to bump or touch a mailbox, he may not discover that it is larger than a fireplug. Parents and teachers who do not verbalize, describe situations, or in any other way bring the visually handicapped child into direct contact with his environment, deprive him of learning opportunities and this deprivation will contribute to later deficiences in mathematics.

As the child progresses through the elementary grades, teachers should assume responsibility for providing as much concretization as possible. Real life experiences and tangible aids should be used to develop abstract concepts and to aid in problem solving. At every step in his education he should be made aware of what seeing children are doing. He should know what ink-print letters, as well as forms, shapes and sizes, look like so that his Braille counterparts will be meaningful.

The visually handicapped child should not be denied the privilege, however limited, of joining in classroom projects. Why should he not be allowed, with the help of a buddy, to participate in a classroom game or help in the classroom store? He needs this, not only for building concepts, but for establishing and maintaining some relationship with his sighted friends.

Since the resource teacher, the classroom teacher, and the parent all have roles in helping the visually handicapped child develop into an educated, self-sufficient, contributing member of society, this manual should serve primarily as an aid to the classroom teacher and the resource teacher as they seek ways to help the child. The remaining portion of this section will suggest important activities for each partner in this most important venture, that of educating visually handicapped pupils.

A. The role of the Braille or resource teacher:

1. To help the pupil adjust to the classroom program.
2. To prepare the classroom mathematics teacher for the presence of a visually handicapped child in her class. The future relationships between the child and the mathematics teacher, the child and his classmates, and between the mathematics and resource teacher depend greatly upon the attitude fostered by this initial contact.

   He should:
   a. discuss the background and ability of the pupil
   b. explain some of the complexities of doing mathematics in Braille
   c. stress the importance of early requests for materials and tests which must be Brailled
   d. describe the specialized materials already available and the procedure for obtaining them
   e. describe tangible aids
   f. determine other specialized materials the teacher will need
   g. assure the teacher that any request for help will be considered conscientiously
3. To maintain liaison with the classroom mathematics teacher by:
   a. fostering a realistic attitude toward pupil's ability to participate in class
   b. helping the classroom teacher in planning pupil's involvement in classroom activities

4. To be familiar with and develop an understanding of the material being taught in the classroom by:
   a. seeking the help of the mathematics teacher for his own understanding
   b. securing copies of mathematics texts or courses of study for his own use

5. To instruct the pupil in the use of the Nemeth Code. Beginning at the elementary level, instruction should parallel the mathematics program in the classroom through:
   a. introducing new symbols as they appear in the printed text or in the classroom
   b. showing the print counterpart of terms or signs used in Braille
   c. clarifying the use of the code in Braille calculations

6. To enrich, reinforce and interpret concepts. A simple explanation by the resource teacher in an individualized situation will sometimes suffice. He should:
   a. use tangible aids already available for purchase
   b. create or modify aids

7. To encourage the pupil and to foster a positive attitude by:
   a. encouraging him to ask for necessary help
   b. seeing that he gets the help he needs
   c. recognizing the importance of his knowing that most books and materials will be available when required
   d. helping him make the necessary adjustments and to develop a positive attitude when materials are not available

B. The role of the classroom teacher:

1. To accept the pupil as an individual entitled to education.

2. To recognize that the visually handicapped pupil must:
   a. learn and use a duel set of symbols, since he must know the Braille symbols along with those used by the sighted
   b. have enough imagination and skill to visualize the mathematical problems in both Braille and ink-print
   c. use aids such as a Braillewriter while in his classroom

3. To modify his requirements and make the goal more realistic for the pupil by:
   a. adapting daily assignment for the visually handicapped pupil
   b. adapting end goals to the ability of the child

4. To become familiar with the tangible aids commonly used by the visually handicapped.

5. To keep in close contact with the resource teacher by:
   a. letting him know when and with what subject matter the child is having trouble
   b. advising the resource teacher as to future lesson plans so that he can collect the necessary aids and prepare the pupil to meet the lesson in the classroom

6. To verbalize more for the benefit of the visually handicapped pupil:
   a. saying aloud all that is being written on the board
   b. stating definitions so that no ambiguity exists or reliance on diagrams is fostered

MATHEMATICAL AIDS AND OTHER DEVICES USED IN COMPUTATION

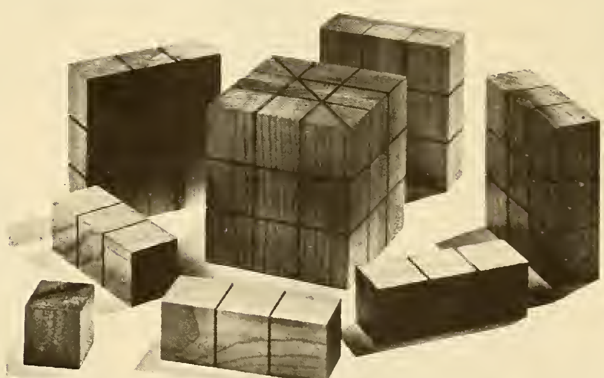
In developing mathematical concepts in visually handicapped pupils, the use of concrete objects and tangible aids cannot be overemphasized. It is the purpose of this section to document some of those aids available for purchase for use at the junior and senior high school level and to illustrate some of the techniques of computation through the use of Braille and non-Braille devices. The aids mentioned are available from the American Printing House for the Blind, 1839 Frankfort Avenue, Louisville, Kentucky 40206; the Howe Press, 175 North Beacon Street, Watertown, Massachusetts 02172; or the American Foundation for the Blind, 15 West 16th Street, New York, New York 10011. Following is a list of a number of mathematical aids and other devices used in computation which are available for purchase.

1. Rulers — 12 in., 15 in.
2. Compass
3. Protractor
4. Graph Paper — Braille and large print
5. Fraction Blocks
5. Fraction Game
7. Tape Measure
8. Master Cube
9. Graphic Aid for Mathematics
10. Mitchell Wire Forms
11. Cranmer Abacus
12. Braillewriters
    a. Perkins
    b. Lavender
13. Braille Slate and Stylus
14. Arithmetic Type-Slates
    a. Brannan Cubarithm
    b. APH Cubarithm or Arabic Numeral
    c. Texas
    d. Taylor
15. Sewell Drawing Board

Teachers should feel free to develop their own aids or to adapt those in their possession in order to fill a particular need.

Several of these devices are designed to illustrate more than one type of mathematical concept.

The Master Cube



The MASTER CUBE is a multipurpose device which can be used to visualize all four operations, or, by using the solid blocks, to stimulate interest in measuring and weighing of volume and capacity.[1] It consists of a four-inch cube, made of polished hardwood, and cut into eight pieces. These pieces are:

1 cube — 1 in.
3 blocks — 3 in. x 1 in. x 1 in.
3 blocks — 3 in. x 3 in. x 1 in.
1 cube — 3 in. x 3 in. x 3 in.

Figures, Drawings and Constructions

The concept of forms and figures is a difficult one for a visually handicapped pupil to grasp. Many teachers, unfortunately, consider this an impossible task. It is difficult, of course, because the child may not have grown up with figures; therefore his appreciation of them may be limited.

Simple geometric forms can be illustrated for the pupil with solid objects or by tracing with the fingers. Here again parents and teachers share the responsibility. As the drawings become more complex, the resource teacher may need to use an already available aid or to develop some other way to illustrate the figures. The MITCHELL WIRE FORMS are a manufactured aid which can be used to illustrate shapes and geometric forms, thus helping the blind child to learn about them.

Actual construction of graphs and figures should not be a necessary requirement of visually handicapped pupils, since many do not have the manual dexterity to perform such operations. However, if they do, the execution of such constructions adds much to the pupils' understanding of concepts, even if the figures are somewhat distorted. There are two available devices which are suggested for this purpose, the Graphic Aid for Mathematics and the Sewell Drawing Board. For those pupils who cannot do construction by any device, the teacher should aim for some oral means of testing the pupil's understanding of graphing and construction procedures.

The GRAPHIC AID FOR MATHEMATICS and its accessories can be used by the visually handicapped pupil with the help of the resource teacher for actual construction practice, or for testing purposes. It has horizontal and vertical lines a half inch apart which the visually handciapped pupil can feel. Points can be set up on a graph with push pins and connected with rubber bands and wires which are provided. With this equipment the pupil and his teacher can also set up circles, triangles, parallel lines, and even bar graphs.

The SEWELL DRAWING BOARD is an upward drawing device which many pupils can learn to use after experience with the Graphic Aid. It can also be used to illustrate the shapes of triangles, rectangles, trapezoids, the position of radii, and diameter, etc.

Other adapted equipment items available for the visually handicapped pupil, and helpful in construction are: the compass, protractor, and ruler. The following are some actual figures that an interested pupil with above average dexterity might try with this equipment:

Parallel lines — 2 units apart

Right angle, right triangle

Two right triangles — 5 units on one side, 4 units on the second side

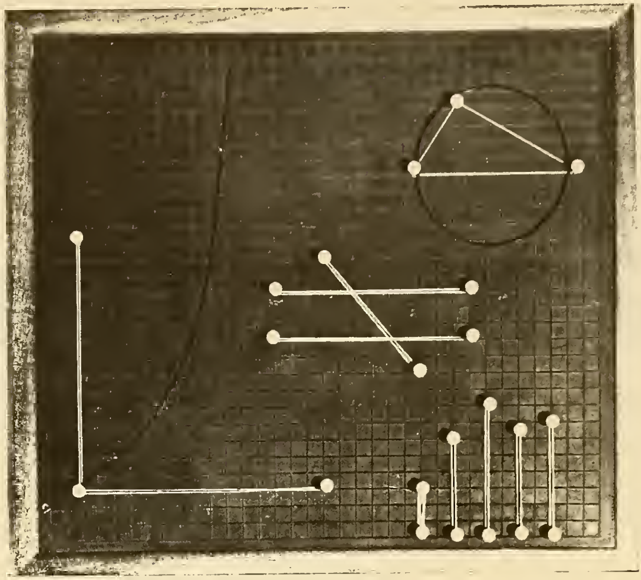
A circle with diameter of three inches

A 2½″ square

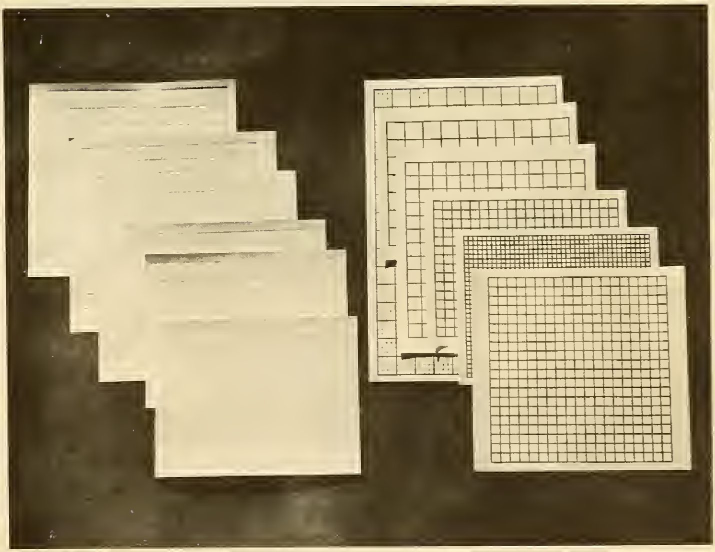
A parallelogram with base of 4″

1. Description adapted from "The Master Cube" — The American Printing House for the Blind, Louisville, Kentucky.

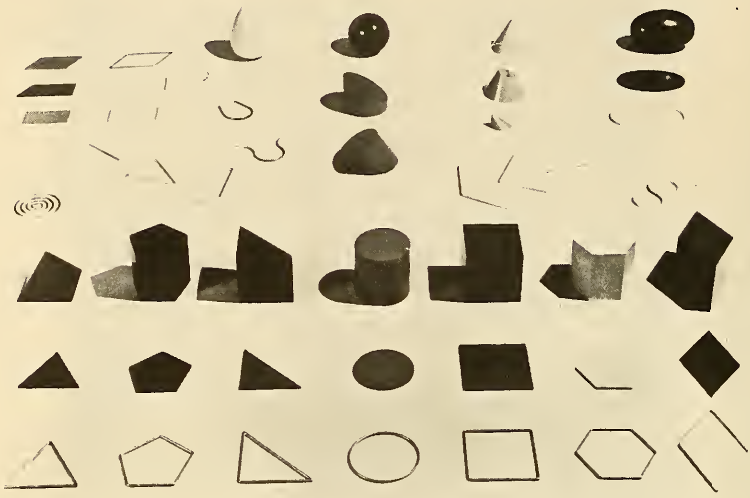
Graphic Aid for Mathematics



Embossed and Bold-Line Graph Sheets



Mitchell Wire Forms



The MITCHELL FORMS consist of a set of geometric figures illustrating outline forms, planes and volumes commonly found in geometric problems.[1] Corresponding figures in all three forms have been built to the same scale (the cube is 3″ x 3″ x 3″), so that the wire frames will fit over the planes of the same shape as well as over the planes of the solids of the same shape. The planes themselves are of the same shape and size as the same planes on corresponding solids. The volumes have been manufactured of hardwood brightly painted in various colors, and the planes made of wood composition painted in colors corresponding to those of their respective volumes. The wire outlines have been fabricated from 1/8″ cadmium-plated steel wire.

The entire set consists of the following items, as shown in the accompanying photograph, and are sold in complete sets only.

15 solids (including 2 parts each for the truncated cylinder and the truncated cone)

11 planes (each 1/8″ thick)

24 wire frames

Computational Devices

Computational devices fall into two broad categories: Braille and non-Braille devices. The Braille devices include the Braillewriters and the slates. Non-Braille devices are the various movable arithmetic type-slates and the abacus.

1. Description adapted from "The Mitchell Forms" — The American Printing House for the Blind, Louisville, Kentucky.

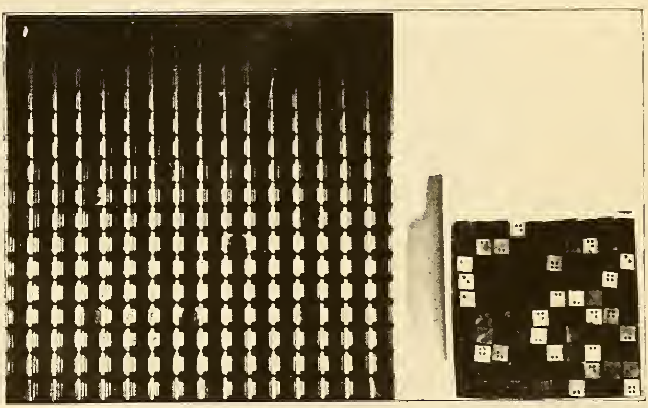
Non-Braille Devices

Arithmetic Type-Slates

The arithmetic type-slates are non-Braille devices used for unwritten computations. These slates consist of square or rectangular plastic, wooden, or metal frames with cells. Three different kinds of type are used, depending upon the shape and size of the cells of the frame, i.e., (1) identical cubes containing Braille digits from 0 to 9 (cube slate) ; (2) elongated square metal or plastic type faced at one end with embossed ink-print arabic numerals; and (3) elongated square metal type faced with two dots at one end and a bar at the other which can be placed in various positions into an 8-sided star-shaped hole so as to give 16 different positions (Taylor slate). By fitting the type into the cells of the frame which they fit, they can be used in calculations of the four basic processes. Arithmetic type-slates present certain limitations:

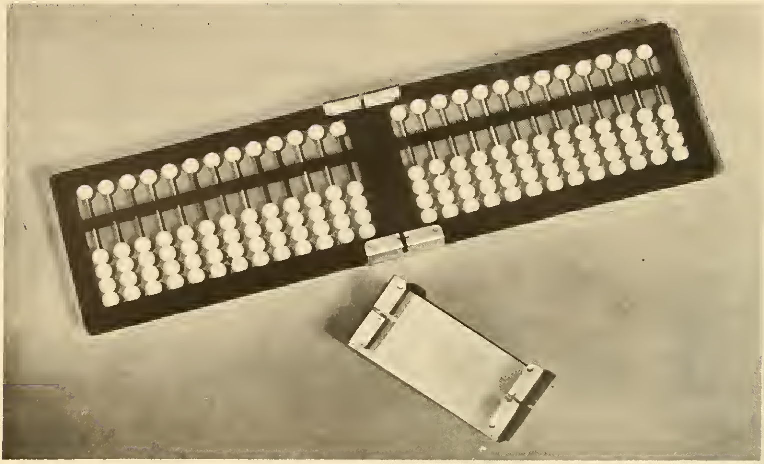
1. the computations are temporary
2. the cubes are small and require finger dexterity
3. there is no provision for using variables
4. some slates require learning a new set of symbols
5. the slates are easily upset and entire calculations destroyed

Brannan Cubarithm Slate



The Cranmer Abacus

The Cranmer Abacus is a self-contained, pocket-sized calculating device with which one can add, subtract, multiply, divide, extract roots, handle decimals, fractions, and trigonometric functions — in short, most processes of arithmetic calculations.[1]. It is a variation of the standard Japanese Soroban, with one bead above the bar and four beads below. This variation has two distinct features that set it apart from other forms of the abacus. First, the operator can read it by touch without fear of upsetting his work. The beads have a small amount of pressure applied to them from behind by a foam-rubber



pad, covered with felt, thus creating a slight friction. Second, adequate bead-spacing and travel-distance allow for easy operation.

A coupling device can be purchased to join two abacuses, thus yielding more columns. This is necessary for many multiplication or division problems.

The abacus presents for the visually handicapped person a quick, quiet, and efficient way to handle all sorts of calculations. Its portability, like the pocket slate, offers a convenient device for both pupils and adults. One can temporarily record telephone numbers, or check the expenditures on a grocery list. Training should begin in elementary grades; it is conceivable that with skill in the abacus may come a confidence in mathematics which will prove most satisfactory when pencil and paper format or organization are not in question or when a record of the calculations is not necessary. The degree to which the abacus is used in junior and senior high school is influenced by the nature of the mathematics program, the ability and grade placement of the student, the flexibility of the mathematics teacher, and the attitude of the pupil. The following description of the use of the abacus will acquaint the reader with the basic principles of its operation. A thorough understanding should be based upon detailed instruction.

Place Value and Operation of the Cranmer Abacus

Each single bead above the lengthwise bar, reading from right to left, represents five of the beads in the column directly below. It may represent five ones, five tens, five hundreds, etc. This device is most helpful in teaching place value. Each bead below the bar represents one of something. What it represents depends upon the column in which it lies. Each bead below the bar also represents ten of those below the bar in the preceding column. A second abacus can be fastened on to represent places to the right of the decimal point (or numbers less than one). It is never necessary to represent any more than nine in any column, for as soon as there are more than nine, ten of them are "cashed in" for one in the next column. A digit such as seven is a symbol which represents that amount — whether it be ones, hundreds, or millions. The unit it represents depends upon the column in which it is placed.

Example 1. To represent the number 36:

Move 3 of the beads in tens column below the separation bar up to the separation bar. This represents three tens or 30. To represent the 6 units move the one bead above the bar in the units column

1. Description adapted from "The Cranmer Abacus for the Blind and Instruction Manuals" — American Printing House for the Blind, Louisville, Ky.

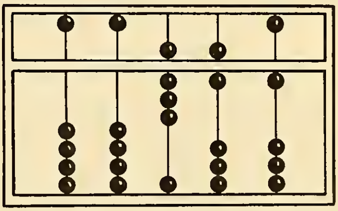
down to the bar and one bead below the bar in the same column up to the bar. The one bead above the bar represents 5 units and the one bead below the bar represents one unit. 5 units and 1 unit are 6 units. You now have 3 tens and 6 units which represents 36.

Example 2. To represent the addition of 847 and 14.

Set the number 847. Begin in the hundreds column. Move the one bead above the bar down to the bar. This represents 5 hundred. Move 3 beads below the bar up to the bar this represents 3 hundreds. 5 hundreds and 3 hundred are 800. Next go to the tens column. Move 4 beads below the bar up to the bar. 4 tens are 40. Next go to the units column. Move the one bead above the bar down to the bar and 2 beads below the bar up to the bar. 5 units and 2 units are 7. You now have 800 + 40 + 7, which is 847.

To add 847 and 14 set the 847 as shown above. Then add one ten to the tens column by moving the 5 bead above the stop down to the bar and pushing the 4 beads below the bar down to the bottom of the tens column. You cannot add 4 units to 7 units because you do not have enough beads left below the bar in the units column. You must move one bead below the bar in the tens column up to the bar. You have now added ten units because one ten is ten ones. This is 6 units more than you wanted to add so you must subtract 6 units from the unit column. You do this by moving the one bead above the bar away from the bar and one of the 2 beads which are against the bottom of the bar away from the bar. You now have one bead above the bar and three beads below the bar in the hundreds column, one bead above the bar and one bead below the bar in the tens column, and one bead below the bar in the units column. This is 861 which is the answer, as shown below.

Notice, that by moving from left to right, you no longer need to be concerned about carrying.

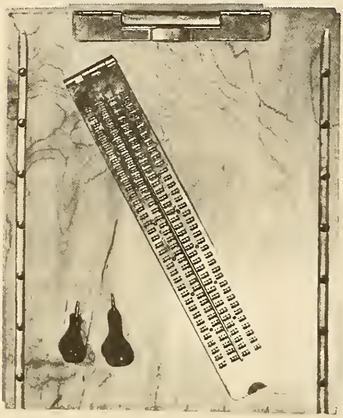


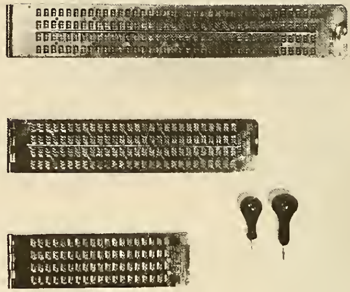
Braille Devices

The Slate and Stylus

The Braille slate[1] is the traditional device for writing Braille by hand. All slates consist of a metal or plastic frame (or guide), which may or may not be designed to be mounted on a solid board. A small pointed steel tool with a handle, called a stylus, is used to punch the Braille dots. Each guide consists of two parts connected at the left end by a hinge. The face of the bottom of the guide is pitted with four lines of a series of six small, round depressions corresponding to the shape and spacing of the dots of the Braille cell. In order to guide the stylus in punching the dots, the top of the frame is punched with four lines of holes that outline the individual Braille cells and correspond to the arrangement of the pits in the bottom of the guide. To write on a slate, paper is inserted between the top and bottom of the guide and is held in place by small pins. The Braille dots are punched downward into the paper, thus making it necessary to write from right to left in order that, when the paper is turned over in position for reading, the Braille characters can be read from left to right.

1. Description is adapted from the American Printing House for the Blind, "Braille Slates and Stylus" — Louisville, Kentucky.





Slates are available in various lengths and some can be carried in a pocket. This offers advantages for short note-taking or writing telephone numbers, for example, but is unsuitable for lengthly computations since the paper must be removed or turned over before the work can be read.

Braillewriters

The Braillewriters are machines which produce standard size Braille. Each machine has six keys which correspond to the dots of the Braille cell ⠿ . The Nemeth Code of Braille Mathematics uses 41 such cells to the line. The examples for this manual are based upon a 29-cell line which causes a more vertical arrangement of some examples. The keys of the Braillewriters are large enough to manipulate easily. The machines are larger, less comfortable to carry and produce more sound than the slates, but they do allow much more flexibility in mathematical calculations. As opposed to the cube slates, the Braillewriters permit the pupil:

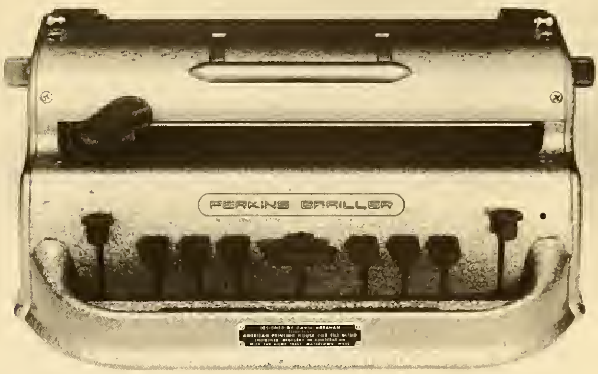
1. to read what he has written
2. to use both letters and numerals in calculations
3. to keep a record of his work
4. to erase or change his calculation
5. to set up some problems in an arrangement similar to the printed problem

The classroom teacher should bear in mind that the visually handicapped pupil must:

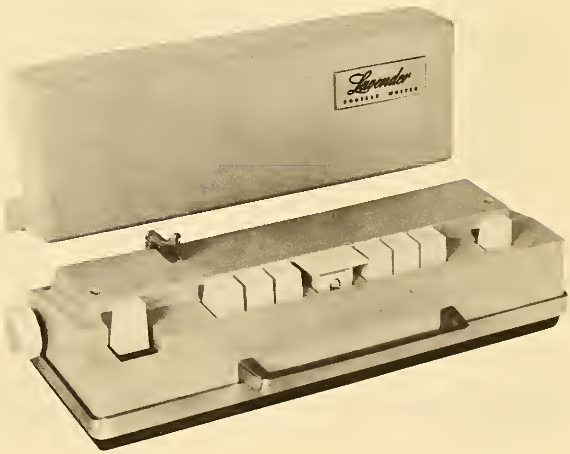
1. have mastered the Braille system of mathematical notation
2. understand the problem and its solution
3. have acquired enough skill to utilize the mental and mechanical processes simultaneously

Many pupils are encouraged to compute mentally. Some resort to mental calculations because they have not acquired enough skill in operation of the device itself. Many can develop these mental shortcuts and maintain accuracy in their calculations. Perhaps this should be one of the criteria for determining to what extent shortcuts are used. The pupil should be reminded, however, that shortcuts are no more than that, and possibly his talents would be used more effectively in solving more difficult problems rather than keeping numbers in his head.

The Perkins Braillewriter



The Lavender Braillewriter



SYMBOLS

Symbols of operation ($+$, —, $\times$, $\div$), symbols of inclusion ([ ], ( ), { } etc.), symbols naming relations ($=$, $\neq$, $\approx$, $>$, $<$), variable or placeholder symbols ($\triangle$, $\square$, —, x, y, t, etc. . . .), and other symbols which indicate or name mathematical ideas or concepts (π, f(z), Σ, etc. . . .) and their Nemeth Code counterparts should be developed as they occur in the mathematics program. In the Nemeth Code the symbols of inclusion are called symbols of grouping and the relation symbols are called signs of comparison. The above mentioned symbols are but a few of the dozens that one might encounter in higher mathematics. A few examples from geometry are the *line* ($\overleftrightarrow{AB}$), the line *segment* ($\overline{AB}$), *ray* ($\overrightarrow{AB}$), the *measure* of (m), the *arc* ($\overset{\frown}{AB}$), the *angle* ($\angle$), or congruency ($\cong$).[1]

Most Frequently Used Nemeth Code Mathematical Indicators

⠼	Numeric	⠸⠹	Mixed Number — Opening
⠸	Punctuation	⠸⠼	Mixed Number — Closing
⠠	Capital	⠜	Radical
⠰	Alphabetic — Roman letter	⠨	Radical — Order — Opening
⠨	Alphabetic — Greek letter	⠨⠻	Radical — Order — Closing
⠸	Type — Boldface and Filled-in Shapes	⠣	Radical — Index
⠨	Type — Italic	⠌	Negation — Slant
⠈	Type — Script	⠳	Negation — Vertical
⠿	Omission	⠰	Subscript
⠪	Cancellation — Opening	⠘	Superscript
⠻	Cancellation — Closing	⠐	Base Line
⠹	Fraction — Simple — Opening	⠐	Multipurpose
⠼	Fraction — Simple — Closing	⠣	Directly Over
⠠⠹	Fraction — Complex — Opening	⠩	Directly Under
⠠⠼	Fraction — Complex — Closing	⠻	Termination

1. In the case of exponents, the position of the symbols is part of the designation. The square root symbol (⠜) and its adaptations are also discussed in this section.

⠠ Enlargement

⠨ Shape Modification

⠫ Shape

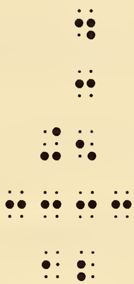
⠈ Superposition

A symbol to the visually handicapped pupil is a dot formation which represents a certain printed *sign.* So that he may interpret accurately what the author or transcriber intends, more than thirty "indicators" have been devised. An indicator is defined in the 1965 NEMETH CODE OF BRAILLE MATHEMATICS as "a symbol for which, with the exception of the cancellation indicators, there is no equivalent sign of print." By the use of indicators, the symbols required in mathematics can be accurately and uniformly interpreted by the visually handicapped pupil. They tell him where things are in relation to each other in print. They tell him when letters are capitalized, that they may be boldface or italicized, that numerals may be above or below the line, etc. They do not, however, tell him how the printed form appears and herein lies the responsibility of his resource teacher.

Symbols of Operation

In the elementary grades the pupil learns the symbols for the signs of operation. His Braille teacher, who has set up his problems both linearly and in column form, should show him how these appear in print. The following exercises illustrate this:

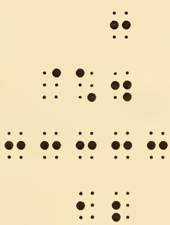
(1)
$$\begin{array}{r} 4 \\ 3 \\ +\ 5 \\ \hline 12 \end{array}$$



(2) $4 + 3 + 5 = 12$

(3) $4 \times 3 = 12$

(4)
$$\begin{array}{r} 3 \\ \times\ 4 \\ \hline 12 \end{array}$$



Symbols of Grouping

When the grouping signs are used in print, their corresponding Braille symbols should be introduced. Parentheses set things apart from other members or group them together. Everything inside can be thought of as one number. $(6 + 2)$ may be thought of as (8). In an expression such as $(4 + 5) \times (7 - 2)$, we would proceed by doing the operation within the symbols first and then finishing the operation. Thus we would have $(9) \times (5) = 45$. The exercise $(7 + 3) - (4 \times 2)$ should be read and interpreted as "the difference between the sum of 7 and 3 and the product of 4 and 2."

Many texts are getting away from using the brackets as symbols of inclusion. The important idea involved in nested symbols of inclusion is that they always appear in pairs. Proceed by removing the inside pair first. Here is an example to show nested symbols of inclusion: $[(3 + 4) - (2 \times 3)] + 15$ It would appear in Braille as follows:

Symbols of Comparison

Relation symbols or signs of comparison ($=$ equal to, $\neq$ not equal to, $\approx$ approximately equal to, $>$ greater than, $<$ less than) also have their Braille counterparts.

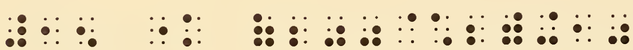
Examples:

(1) $(4 \times 6) - (14 \div 2) = 17$ Read as "the difference between the product of 4 and 6 and the quotient of 14 and 2 equals 17."

(2) $(9 + 2) \times (14 - 8) \neq 50$ Read as "the product of the sum of 9 and 2 and the difference between 14 and 8 is not equal to 50."

(3) $5 + (17 - 2) > 6$ Read as "the sum of 5 and the difference between 17 and 2 is greater than 6."

(4) $15 < (200 \times 2) + 10$ Read as "15 is less than the sum of the product of 200 and 2 and 10."



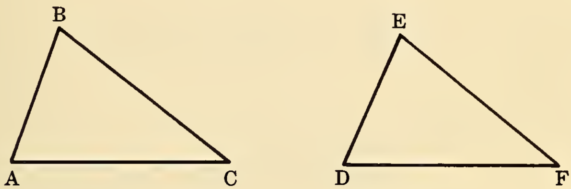
(5) $\sqrt{3} \approx 1.732$ Read as "the square root of 3 is approximately equal to 1.732."

Placeholder Symbols

Placeholder symbols such as ▭, n, y, t, x, etc. . . . are used to represent a number. If these print symbols are in the form of a shape, the visually handicapped pupil must know the indicator. For example, in the exercise: $\square = 75 - (10 \times 5)$, the Braille version is:

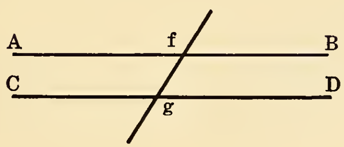
Examples of use of some geometric symbols:

(1)



	Read as:
Given: $\triangle$ ABC and $\triangle$ DEF	triangle ABC and triangle DEF
$\overline{AB} \cong \overline{DE}$	segment AB is congruent to segment DE
$m\angle A = m\angle D$	the measure of angle A is equal to the measure of angle D

(2)



	Read as:
Given: AB $\parallel$ CD	lines AB and CD are parallel
$\angle f \cong \angle g$	angle f is congruent to angle g

Exponents

An exponent tells how many times a number is used as a factor. It is placed to the right of and above the base numeral. Thus 5^2 is read as "5 to the second power" or "5 squared." This must be clarified for the visually handicapped because the numeral (^{2}) which denotes the power is written in Braille on the same line as the base numeral. A Braille indicator is used to indicate that the 2 is an exponent. 5^2 is written in Braille in the following manner: ⠼⠢⠘⠆ . Notice that between the 5 and 2 is the Braille indicator, ⠘ , which prohibits the interpretation 52.

Here are a few more examples to illustrate meaning of exponents in powers.

(1) $5^2 = 5 \times 5 = 25$ — 5 is the base number and 2 is the exponent, the entire expression 5^2 is the power

(2) $5^3 = 5 \times 5 \times 5 = 125$ — Read as: "5 to the third power" or "5 cubed"

(3) $5^4 = 5 \times 5 \times 5 \times 5 = 625$ — Read as: "5 to the fourth power"

(4) In algebra, a variable such as x, y, or z, might be the base number.

$x^2 = x \cdot x$

The multiplication dot sometimes used in print has a corresponding sign in Braille. Examples:

$x^3 = x \cdot x \cdot x$

d • y in Braille is: ⠙⠡⠽

$x^5 = x \cdot x \cdot x \cdot x \cdot x$

while d $\times$ y is: ⠙⠈⠡⠽

(5) In simplying an example such as $6y^4$, the power of y is found before multiplying by 6.

To find $6y^4$ when y is 2:

(*correct*) $6y^4 = 6\ (2^4) = 6\ (16) = 96$

⠼⠖⠽⠘⠲ ⠨⠅ ⠼⠖⠷⠆⠘⠲⠐⠾ ⠨⠅ ⠼⠖⠷⠂⠖⠾
⠨⠅ ⠼⠔⠖

(*incorrect*) $6y^4 = (6 \times 2)^4 = 12^4 = 12 \times 12 \times 12 \times 12$

(6) Find $3x^2y^4$ when x is 5 and y is 1

$3x^2y^4 = 3\ (5^2 \times 1^4) = 3\ (25 \times 1) = 75$

Roots, Radicals and Indices

Braille symbols for problems involving roots, radicals and indices are as complicated as the problem itself. The rules for transcribing radicals should be explained to the pupil as they are presented. The indicators which do not appear in print should be explained and he should be shown how they interpret the printed form. The radical sign $\sqrt{}$ is used to indicate the root of a number and is represented by the Braille symbol ⠜ . The radicand is the number under the sign $\sqrt{}$. Sometimes the word radical refers to the sign and the number together . . . $\sqrt{8}$. The index number indicates the root that is to be taken and is placed in the "pocket" of the radical sign: $\sqrt[3]{}$. To read this exercise, the visually handicapped pupil must learn an index indicator, a radical indicator, and a termination indicator. Thus:

$\sqrt[3]{8}$ is read "the cube root of 8"

⠣ ⠒ ⠜ ⠦ ⠻

If no index is present, tradition indicates that the square root is required.

Note the labeling of the following radical: $\sqrt[3]{64}$ index radicand radical sign

$\sqrt{64}$ indicates the square root of 64.
The omission of the index indicates
the square root.

$\sqrt[3]{64}$ indicates the cube root of 64. The index 3 must be present.

As radicals become more involved, additional indicators are used. Sometimes radicals occur under radicals. They usually are simplified from the inside out. An example like $\sqrt[3]{x^2 + \sqrt[3]{x^2 + y^2} + y^2}$ might occur in analytic geometry. To simplify this when x is 5 and y is 3, work on the inside first:

$$
\begin{aligned}
\sqrt[3]{x^2 + \sqrt[3]{x^2 + y^2} + y^2} &= \sqrt[3]{5^2 + \sqrt[3]{5^2 + 3^2} + 3^2} \\
&= \sqrt[3]{25 + \sqrt[3]{25 + 9} + 9} \\
&= \sqrt[3]{34 + \sqrt[3]{34}} \\
&\approx \sqrt[3]{34 + 3.239} \\
&\approx \sqrt[3]{37.239} \\
&\approx 3.340
\end{aligned}
$$

Fractions

Reading and Writing Fractions

The concept of fractions presents special problems for the visually handicapped pupil. The resource teacher should be ready to enrich or reinforce with tangible aids any area in which the pupil shows a lack of understanding. It should not be assumed that he has had enough background experience to assure a complete understanding of fractions. The classroom teacher should recognize the difficulties involved in the reading, writing and solving of problems in Braille. He should make every effort to verbalize as much as possible, recognizing that some terms may be represented differently in Braille.

Wooden fraction blocks which illustrate a whole, halves, fourths, and eighths are available to help the visually handicapped pupil understand the principle.

The Sewell slate or a spatial arrangement on the braillewriter will help the pupil to see the fraction as his sighted friends see it. Usually fractions are written linearly in Braille and read from left to right. What the sighted person sees as $\frac{2}{3}$ with numerals written above and below the fraction line, the visually handicapped sees written on one line as: 
Thus, to read a fraction, the visually handicapped pupil first sees the Braille indicator  which precedes a fraction. This is followed by a Braille symbol for the number which is the numerator of the fraction, the Braille symbol  which indicates the fraction line, a symbol for the number which is the denominator of the fraction, and finally the Braille indicator  which marks the end of the fraction.

Complex fractions are fractions which have fractions as numerators or denominators and the complexity of the Braille representation increases, as illustrated:

(1) for $\dfrac{\frac{7}{8}}{6}$ 

(2) for $\dfrac{\frac{3}{4}}{\frac{7}{8}}$ 

(3) for $\dfrac{3 - \frac{x}{y}}{\frac{2x}{y}}$ 

The Slash

The visually handicapped pupil should also be aware of the use of the slash mark as opposed to the horizontal fraction line. The rules governing the transcription of the slash are intended to permit the blind reader to extract the same meaning which the sighted reader extracts. The rules are also devised to keep the reader aware of the distinction between the horizontal and diagonal fraction lines.

The Solution of Equations

The emphasis of much of the secondary school program in mathematics is directed toward the solution of equations as a tool in problem solving, as a method of developing understanding of a mathematical system, and as a skill needed to function in today's culture. Different teachers and different texts present these in different ways. All methods are based on the use of fundamental principles which are frequently called "properties of equality" or the "fundamental axioms." The classroom teacher and the resource teacher should attempt to use the same technique so that the visually handicapped pupil is not confused by a multiplicity of methods.

An equation basically states that two forms of a number or two mathematical expressions name the same number. Thus $2 + 6$ and $4 + 4$ may be the two members of an equation since $2 + 6$ and $4 + 4$ both name 8. If $3 + x$ and 5 are used as the two members of an equation, the "solution of the equation" becomes the process of determining what number or numbers can be used to replace the x so that the resulting number sentence is true.

The process of solution is one of transforming a complex equation into a simpler one whose "root" or "solution" is obvious. The transformations possible are based on the fact that the two members of the equation are numbers, and operating on a number with another number always yields the same result. The same basic properties are the four basic operations of arithmetic and may be characterized as follows:

If $a = b$, then $a + c = b + c$. The addition property of equality.

If $a = b$, then $a - c = b - c$. The subtraction property of equality.

If $a = b$ then $ac = bc$. The multiplication property of equality.

If $a = b$ then $\frac{a}{c} = \frac{b}{c}$ $(c \neq o)$. The division property of equality.

Sometimes mathematics teachers use a standard format for using these properties and the visually handicapped child may have difficulty seeing the relation and the property involved.

In one mathematics class the equation $3x - 2 = 10$ may be solved using the format:

$$\begin{array}{rcl} 3x - 2 & = & 10 \\ +2 & & +2 \\ \hline 3x & = & 12 \\ \frac{3x}{3} & = & \frac{12}{3} \\ x & = & 4 \end{array}$$

A preferred solution is as follows:

$3x - 2 = 10$

$3x = 12$ applying the addition property of equality and adding 2 to $3x - 2$ and 2 to 10.

$x = 4$ applying the division property and dividing 3x by 3 and 12 by 3.

The solution is not yet complete. The pupil must either rebuild the original equation by "reversing" the process or check to see if $3x - 2$ does yield 10 when the variable is replaced with 4.

Example:

$5x - 2 = 3x + 4$		
$2x - 2 = 4$		subtracting 3x from both sides
$2x = 6$		adding 2 to both sides
$x = 3$		dividing both sides by 2
$5(3) - 2$	$3(3) + 4$	substituting 3 for x
$15 - 2$		
13	$9 + 4$	in both members and
	13	doing the arithmetic

and since both expressions yield 13

3 is a solution or root of $5x - 2 = 3x + 4$.

To solve some algebraic equations such as $\frac{a}{4} = \frac{9}{12}$, a short cut of the multiplication property of equality, cross multiplication, can be used. Multiplying the a by 12 and the 9 by 4 is really multiplying both sides by the least common denominator, 12. This yields the equation $12a = 36$ which is transformed by applying the division principle and obtaining $a = 3$ which has the obvious solution, 3.

The quadratic equations such as $3x^2 + 4x - 2 = 4 - 3x$ of elementary algebra are solved using the same fundamental properties with one additional. The quadratic equation is transformed by the properties of equality already mentioned so that one member is 0. The other member which is an algebraic expression having a quadratic term is then factored and expressed as a product. A method for simplifying the quadratic to two linear equations can be developed since the product is 0 and one or both factors must be 0. Straight-forward techniques for solving quadratic equations, "completing the square" and "the formula", are developed from this application.

Example:

$3x^2 + 4x - 2 = 4 - 3x$		
$3x^2 + 7x - 6 = 0$		Subtracting $4 - 3x$ from both sides
$(3x - 2)(x + 3) = 0$		Factoring the left member
$3x - 2 = 0$ or $x + 3 = 0$		Applying, If $ab = 0$, then $a = 0$ or $b = 0$
$3x = 2$ $x = -3$		Solving the linear equations
$x = \frac{2}{3}$		
$3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) - 2$	$4 - 3$	Replacing the
$3\left(\frac{4}{9}\right) + 4\left(\frac{2}{3}\right) - 2$	$4 - 2$	"x" with $\frac{2}{3}$ and
$\frac{4}{3} + \frac{8}{3} - \frac{6}{3}$	2	doing the computation
$\frac{6}{3}$		yields $\frac{6}{3} = 2$

$3(-3)^2 + 4(-3) - 2$	$4 - 3(-3)$	replacing the
$3(9) - 12 - 2$	$4 + 9$	x with -3 and
$27 - 14$	13	doing the computation
13		yields $13 = 13$

Both $\frac{2}{3}$ and -3 are roots of the equation $3x^2 + 4x - 2 = 4 - 3x$

Fractional Expressions and Their Interpretations

The following section is not intended to serve as a complete foundation course for the resource teacher, but rather as a refresher. Other help may be obtained from examples in the pupil text, and from the classroom teacher.

Some of the examples in this section have been illustrated in Braille for the reader's convenience.

The following topics, which have to do with numerical and algebraic fractions, are covered in this section:

Factoring

Exponents in multiplication of algebraic fractions

Exponents in division of algebraic fractions

Simplifying (or reducing)

Multiplication of fractions

Division of fractions

Addition of fractions

Subtraction of fractions

Factoring

Factoring is an important operation throughout mathematics. It is used in arithmetic to simplify fractions, in algebra for simplifying expressions and solving quadratic equations, in trigonometry for simplifying identities. The process is separating a number (product) into the numbers (factors) used to obtain the product by multiplication. Since 2×3 is 6 the 2 and 3 are factors of 6. The factors of a number which are most interesting are its prime factors. These are factors which are prime numbers. The prime factors of 12 are 2, 2, and 3.

Exponents in Multiplication

To multiply two powers of the same number the exponents are added.

$x^3 = x \cdot x \cdot x$ and $x^5 = x \cdot x \cdot x \cdot x \cdot x$ so $(x^3)(x^5)$

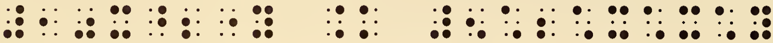
$= (x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x \cdot x)$ which is $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$

or x^8

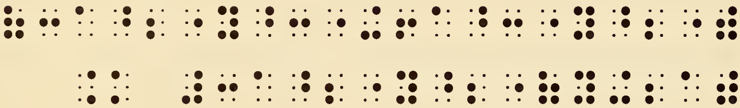
In general, $x^m \cdot x^n = x^{m+n}$

(1) $5cd = 5 \cdot c \cdot d$

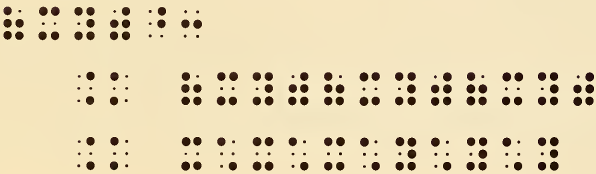
(2) $10x^2y = 5 \cdot 2 \cdot x \cdot x \cdot y$



(3) $(3a^2y^3 + 6a^3y^2) = 3a^2y^2(y + 2a)$



(4) $(xy)^3 = (xy)(xy)(xy) = x \cdot x \cdot x \cdot y \cdot y \cdot y$

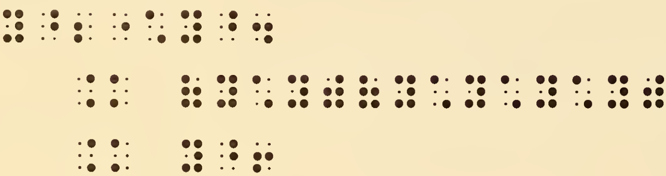


(5) $x^2 + 7x + 10 = (x + 2)(x + 5)$

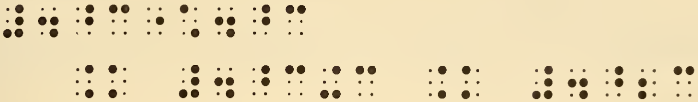
To multiply two powers of the same number the exponents are added.

Examples:

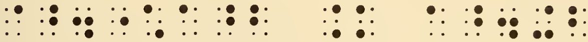
(1) $y^2 \cdot y^4 = (y \cdot y)(y \cdot y \cdot y \cdot y) = y^6$



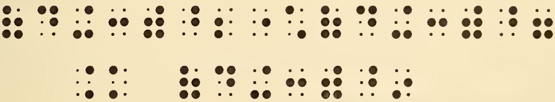
(2) $4^c \cdot 4^c = 4^{c+c} = 4^{2c}$



(3) $a^4 \cdot a^b = a^{4+b}$



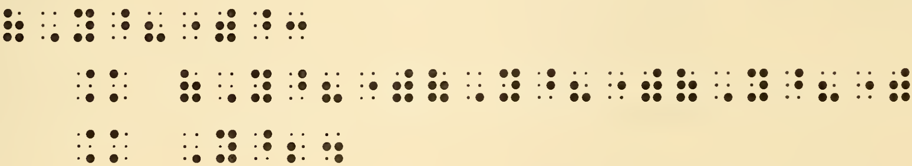
(4) $(d + 3)^2 \cdot (d + 3)^7 = (d + 3)^9$



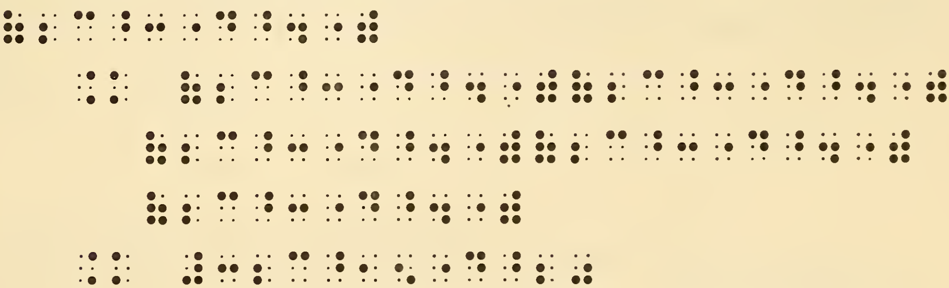
To raise a monomial to a power, each factor of the monomial must be raised to that power. $(b^3)^2$ means use (b^3) as a factor twice, or $(b^3)(b^3)$ which yield b^6. In general $(x^m)^n$ is x^{mn}.

Examples:

(1) $(y^8)^3 = (y^8)(y^8)(y^8) = y^{24}$



(2) $(2c^3d^4)^5 = (2c^3d^4)(2c^3d^4)(2c^3d^4)(2c^3d^4)(2c^3d^4) = 32c^{15}d^{20}$



Exponents in Division

To find the quotient of two powers of the same number, the exponent of the divisor is subtracted from the exponent of the dividend.

$$c^6 \div c^2 = c^{6-2} = c^4$$

or $$\frac{c^6}{c^2} = \frac{c \cdot c \cdot c \cdot c \cdot c \cdot c}{c \cdot c} = c^4$$

in general $$\frac{c^m}{c^n} = c^{m-n}$$

Examples:

(1) $2^8 \div 2^5 = 2^{8-5} = 2^3$

(2) $f^7 \div b^3 = b^{7-3} = b^4$

Multiplying Fractions

The product of two fractions is the product of the numerators divided by the product of the denominators.

(1) $\frac{3}{5} \times \frac{2}{7} = \frac{3 \times 2}{5 \times 7}$ or $\frac{6}{35}$

(2) $$\frac{3x^2y}{2xy^2} - \frac{4xyz}{9x^2yz} = \frac{12x^3y^2z}{18x^3y^3z}$$

$$= \frac{6 \cdot 2 \cdot x^3 \cdot y^2 \cdot z}{6 \cdot 3 \cdot x^3 \cdot y^2 \cdot y \cdot z}$$

$$= \frac{3y}{2}$$

note $\frac{6}{6}$, $\frac{x^3}{x^3}$, $\frac{y^2}{y^2}$, $\frac{z}{z}$ each equal 1

and since 1 times any number equals that number, $\frac{12x^3y^2z}{18x^3y^3z}$ simplifies to $\frac{2}{3y}$.

This process, sometimes called cancellation, is used to simpify or "reduce" fractions to lowest terms.

The simplification process can be performed before the fractions are multiplied. For example:

$$\frac{3}{5} \times \frac{10}{21} = \frac{3}{5} \times \frac{2 \cdot 5}{3 \cdot 7}$$

$$= \frac{2}{7}$$

Combining knowledge of multiplication and factoring makes it possible to multiply complicated algebraic fractions with ease.

$$\frac{a^2 + 7ab + 10b^2}{a^2 + 6ab + 5b^2} \cdot \frac{a + b}{a^2 + 4ab + 4b^2} =$$

$$\frac{(a + 2b)(a + 5b)}{(a + b)(a + 5b)} \cdot \frac{a + b}{(a + 2b)(a + 2b)} = \frac{1}{a + 2b}$$

$$\frac{a^2 + 7ab + 10b^2}{a^2 + 6ab + 5b^2} \cdot \frac{a + b}{a^2 + 4ab + 4b^2} \cdot \frac{a + 2b}{1}$$

$$\frac{\overset{1}{\cancel{(a + 2b)}}\,\overset{1}{\cancel{(a + 5b)}}}{\underset{1}{\cancel{(a + b)}}\,\underset{1}{\cancel{(a + 5b)}}} \cdot \frac{\overset{1}{\cancel{(a + 2b)}}}{\underset{1}{\cancel{(a + 2b)}}\,\underset{1}{\cancel{(a + 2b)}}} \cdot \frac{a + b}{1} = 1$$

Examples:

(1) $\frac{4}{5} \div \frac{3}{4}$

$$\frac{4}{5} \times \frac{4}{3} = \frac{16}{15} = 1\frac{1}{15}$$

(2) $\frac{14}{15} \div \frac{18}{21}$

$$\frac{14}{15} \times \frac{21}{18}$$

$$\frac{\overset{1}{\cancel{2}} \cdot 7 \cdot \overset{1}{\cancel{3}} \cdot 7}{\underset{1}{\cancel{3}} \cdot 5 \cdot 3 \cdot \underset{3}{\cancel{6}}} = \frac{49}{45} = 1\frac{4}{45}$$

(3) $\frac{y}{4} \div \frac{y}{12}$

$$\frac{\overset{1}{y}}{\underset{1}{4}} \times \frac{\overset{3}{12}}{\underset{1}{y}} = 3$$

(4) $\frac{a^2 - b^2}{x^2 - y^2} \div \frac{a + b}{x - y}$

$$\frac{\overset{1}{\cancel{(a + b)}}\,(a - b)}{(x + y)\,\underset{1}{\cancel{(x - y)}}} \times \frac{\overset{1}{\cancel{x - y}}}{\underset{1}{\cancel{a + b}}} = \frac{a - b}{x + y}$$

Division of Fractions

The division of fractions is frequently taught as the trick of inverting the divisor and multiplying. An explanation of the derivation of this "rule" is dependent on the relation of multiplication and division.

Since $a \div b = c$ means $c \cdot b = a$

$$\frac{2}{3} \div \frac{5}{7} = ? \text{ means } ? \times \frac{5}{7} = \frac{2}{3}$$

? is the quotient or answer to the division problem.

In order to get the left side of the equation to become "?", it must be

multiplied by $\frac{7}{5}$ (the reciprocal of $\frac{5}{7}$)

To maintain the equality of the relation, the $\frac{2}{3}$ must also be multiplied by the $\frac{7}{5}$.

Thus $? \times \frac{5}{7} = \frac{2}{3}$

becomes $? \times \frac{5}{7} \times \frac{7}{5} = \frac{2}{3} \times \frac{7}{5}$ and

since $\frac{5}{7} \times \frac{7}{5} = 1$ and $? \times 1 = ?$ the equation becomes

$? = \frac{2}{3} \times \frac{7}{5}$.

Following this process with the general form for two fractions, $\frac{a}{b} \div \frac{c}{d}$ yields $\frac{a}{b} \times \frac{d}{c}$ as the pattern for dividing two fractions. Thus it always happens that the answer to a division of two fractions can be obtained by multiplying the dividend by the reciprocal of the divisor.

Addition and Subtraction of Fractions

Fractions must be expressed with the same number in the denominator in order to add or subtract them. If the denominators are not the same, the procedure to use depends on the multiplication property of 1 to change the fractions so that they do have the same number in the denominator.

To add $\frac{2}{3}$ and $\frac{4}{5}$ the $\frac{2}{3}$ is changed by multiplying $\frac{2}{3}$ by 1 in the form of $\frac{5}{5}$ to obtain $\frac{10}{15}$ and the $\frac{4}{5}$ is multiplied by $\frac{3}{3}$ to obtain $\frac{12}{15}$. Thus,

$$\frac{2}{3} + \frac{4}{5} = \frac{10}{15} + \frac{12}{15}$$

$$= \frac{22}{15}$$

Examples:

(1)

$$\begin{array}{rcl} \frac{4}{5} & \frac{4}{5} & = \frac{8}{10} \\ + \frac{1}{2} & \frac{1}{2} & = \frac{5}{10} \\ & & = \frac{13}{10} \text{ or } 1\frac{3}{10} \end{array}$$

(2)

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd}$$

$$= \frac{ad + bc}{bd}$$

(3) $\frac{a}{xy}+\frac{a}{yz}=\frac{az}{xyz}+\frac{ax}{xy}$

$$=\frac{az+ax}{xyz}$$

(4) $\frac{3}{4}-\frac{1}{3}=\frac{9}{12}-\frac{4}{12}$

$$=\frac{5}{12}$$

(5) $\frac{3a}{c}-\frac{2a}{d}=\frac{3ad}{cd}-\frac{2ac}{cd}$

$$=\frac{3ad-2ac}{cd}$$

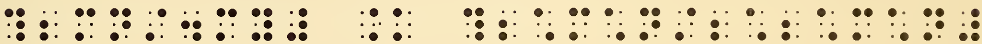
Fraction Indicators

In working with fractions, the pupil will have to know the use of the indicators: cancellation , termination , base line , superscript , and subscript .

The following examples show the use of these indicators.

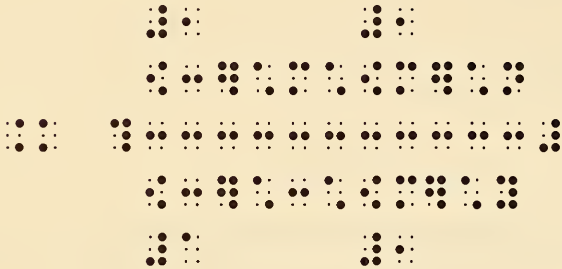
(1) $\frac{mn}{my}=\frac{n}{y}$ 

(2) $\frac{2mn}{4my}=\frac{2\cdot m\cdot n}{2\cdot 2\cdot m\cdot n}=\frac{n}{2y}$



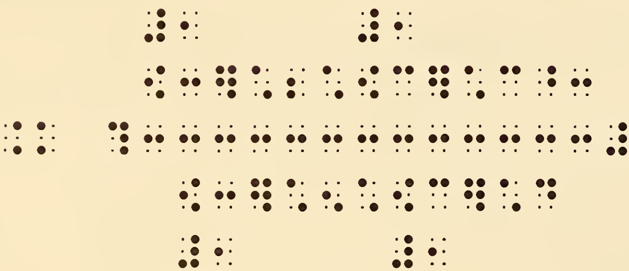
(3)

$$\frac{3m^2n}{9my} = \frac{\overset{1}{\cancel{3}} \cdot m \cdot \overset{1}{\cancel{m}} \cdot n}{\underset{1}{\cancel{3}} \cdot 3 \cdot \underset{1}{\cancel{m}} \cdot y} = \frac{1mn}{3y}$$



(4)

$$\frac{6c^4}{15cd} = \frac{\overset{1}{\cancel{3}} \cdot 2 \cdot \overset{1}{\cancel{c}} \cdot c}{\underset{1}{\cancel{3}} \cdot 5 \cdot \underset{1}{\cancel{c}} \cdot d} = \frac{2c^3}{5d}$$



(5) $$\frac{6}{15}=\frac{3\cdot 2}{3\cdot 5}=\frac{2}{5}$$

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⠹⠖⠌⠂⠢⠼ ⠨⠅ ⠹⠒⠡⠆⠌⠒⠡⠢⠼
    ⠨⠅ ⠹⠆⠌⠢⠼
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(6) $$\frac{6c}{15}=\frac{\overset{1}{\cancel{3}}\cdot 2\cdot c}{\underset{1}{\cancel{3}}\cdot 5}=\frac{2c}{5}$$

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⠹⠖⠉⠌⠂⠢⠼
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    ⠼⠂
    ⠪⠒⠻⠡⠆⠡⠉
⠨⠅ ⠹⠒⠒⠒⠒⠒⠒⠒⠼ ⠨⠅  ⠹⠆⠉⠌⠢⠼
     ⠪⠒⠻⠡⠢
     ⠼⠂
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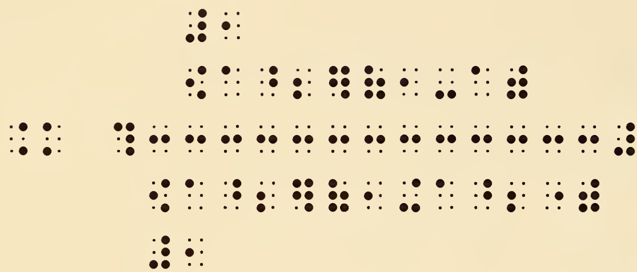
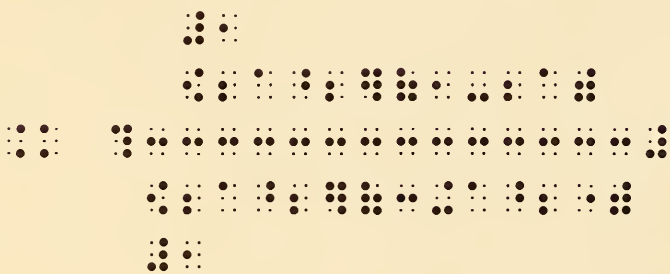
(7)

$$\frac{a^2 - a^3}{a^2 + a^4} = \frac{\overset{1}{\cancel{a^2}}(1 - a)}{\underset{1}{\cancel{a^2}}(1 + a^2)} = \frac{1 - a}{1 + a^2}$$



(8)

$$\frac{2a^2 - 4a^3}{6a^2 + 2a^4} = \frac{\overset{1}{\cancel{2a^2}}(1 - 2a)}{\underset{1}{\cancel{2a^2}}(3 + a^2)} = \frac{1 - 2a}{3 + a^2}$$



MEASUREMENT

The concept of measure and the idea of indicating size using numbers is particularly difficult for the visually handicapped pupil because he has not seen the relation of "greater than" or "smaller than" in early childhood experiences. He doesn't "see" the 100 yards of a football field, the 90 feet from home plate to first base. These must all be learned in the abstract.

Many measurement concepts are based upon what a person sees. The seeing child comes into contact daily with rulers, yardsticks, tape measures, speedometers, blocks, quarts of milk, etc. He compares lengths, widths, sizes and shapes. He is bombarded with measurement concepts without knowing it. The visually handicapped child, however, must be put into contact with these things. He must be told and shown the relation of an inch to a foot and that his room is 9 ft. by 12 ft. He must be told that the distance from home to school is two miles, or that he lives one mile from his friend. He must be told as he rides that he has traveled one mile, two miles, etc., and that he is going 35 m.p.h. His skill in mobility is based upon his ability to judge distances. If he doesn't ask, and if no one tells him these things, he will never be able to fill the gap which will determine his ability to grasp other measurement concepts.

The parent, the elementary teacher, the resource teacher, and the mathematics teacher each has an important role to play in helping the visually handicapped child develop understanding of the concept of measurement and facility in making measures. The visually handicapped pupil must have "hands on" teaching — he cannot be taught to measure by a "show and tell" routine in which he imitates the teacher.

It is essential that all children, including the visually handicapped, learn that a measurement is a comparison by number of the relative size of an object to a standard size of the same object. For example, a segment measuring 6″ in length is six times as long as a segment measuring 1″ in length. The visually handicapped child should also have some notion of the fact that all measurements are approximate. Efforts should be made by both the mathematics teacher and the resource teacher to help the child develop an awareness of the distinction between "accuracy" and "precision."

Length, area, volume, capacity, and weight are the usual types of measurements made in school. Children should learn about the basic units in which these are measured, and have an understanding of how the methods of calculating various measurements are derived.

Linear Measure

In learning about linear measure, the pupil becomes acquainted with such things as the braille ruler or tape measure. He will need a method for showing a stopping place when measuring something longer than his instrument. A piece of tape may be used as a marker. The Graphic Aid can be used to construct line segments of various lengths. The plane figures in the Mitchell collection are good equipment for practice of linear measurement. The pupil can measure the heights and lengths of furniture at home and school. He can measure books, pencils, chairs, doorways, etc. He can compare the lengths of his fingers, hands, arm, and arm span with the other items measured.

Square Measure (Area)

The basic unit of area is the square unit which is the surface enclosed by a square 1 unit on a side. Basically, the measurement of the area of a surface is the number of these units it takes to cover the surface being measured. To derive the formula "A" = "l • w" for the area enclosed by a rectangle, the number of square units in one row along the base of the rectangular boundary is the same as the number of units in linear measure in the length of the boundary segment. The number of such rows is equivalent to the number of units of linear measure in the "width" of the rectangle. Thus there are "w" rows of "l" units making a total of "l • w" units of area.

The formula for the area of the other common plane figures is derived from this basic formula.

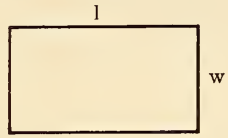
Cubic Measure (Volume)

Volume is measured by counting the number of cubic units which it takes to "fill" a solid. The basic formula again multiplies the number of units in a row along one edge by the number of such rows on a layer to determine the number of units in a layer. This product is multiplied by the number of layers to get the total number of units. The formula $V = l \cdot w \cdot h$ is used for prisms.

A variation of this formula uses $l \cdot w$ as the area of the base. This variation gives a convenient method for calculating the volume of the triangular, octagonal and hexagonal prism, or prisms with other shaped bases. This formula is also convenient for use with cylinders for the area of the base is the area of the circular base. In the mathematics program of the secondary school little or no attention is paid to the volume of cylinders which are not circular.

Basic Geometric Figures and Their Associated Formulas

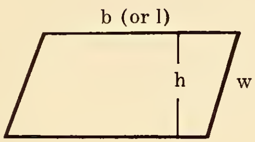
Rectangle



$A = lw$

Area equals length times width.

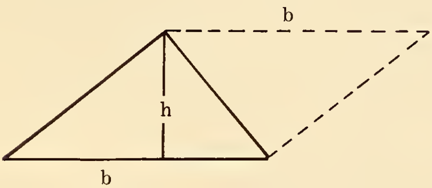
Parallelogram



$A = bh$

The width is a different measurement from the height. The height is the number to be used in finding the area of a parallelogram.

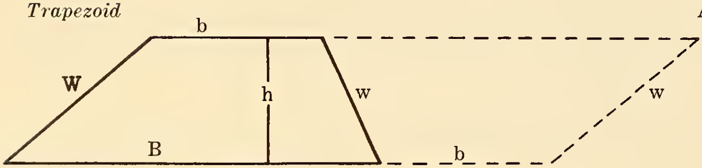
Triangle



$A = \frac{1}{2} bh$

Each triangle is half of some parallelogram. This fact becomes obvious if the triangle is duplicated and the two placed properly.

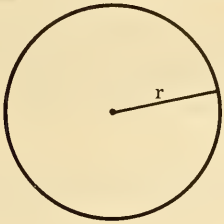
Trapezoid



$A = \frac{1}{2} h (B + b)$

Each trapezoid is half the area of the parallelogram formed by duplicating and placing the two together properly.

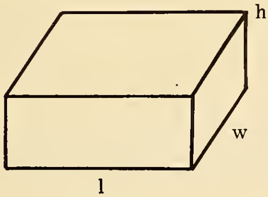
Circle



$A = \pi r^2 \quad (\pi^1 = 3.14)$

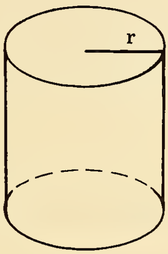
For the derivation of this formula see Peters and Schaaf MATHEMATICS, A MODERN APPROACH, Book 2. Van Nostrand, Princeton, N. J. 1965, p. 262-263.

Rectangular Prism



$v = l\,w\,h$

Circular Cylinder



$v = \pi\, r^2\, h.$

Time, Rate, and Distance

Problems dealing with distance are based on the formula:

$$\text{Distance} = \text{rate} \times \text{time}$$

or $$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

or $$\text{Time} = \frac{\text{distance}}{\text{rate}}$$

The arrangement of the equation depends upon which 2 items of information are given. If the visually handicapped person has developed a kind of mental measurement of time and a feeling for the speed of a vehicle, it should not be too difficult to develop some concept of distance covered on the basis of time and speed. Trips made regularly by the pupil offer the best illustrations.

Examples:

(1) John rides to his cousin's house in a taxi which goes 30 m.p.h. for 2 hours. How many miles away does his cousin live?

$$\text{Distance} = \text{rate} \times \text{time}$$

$$D = rt$$

$$D = 30 \times 2$$

$$D = 60$$

John's cousin lives 60 miles away.

(2) A jet plane traveling 550 m.p.h. leaves the Cleveland airport at 2 P.M. and travels 1375 miles to a city in Texas. How long does it take the plane to reach its destination in Texas?

$$\text{Time} = \frac{\text{distance}}{\text{rate}}$$

$$T = \frac{d}{r}$$

$$T = \frac{1375}{550}$$

$$T = 2\frac{1}{2}$$

The plane reaches its destination in $2\frac{1}{2}$ hours.

(3) John rides 5 miles to school in a taxi in 12 minutes. How fast does the taxi travel?

Note: 12 min. $= \frac{1}{5}$ hour

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

$$r = \frac{d}{t}$$

$$r = \frac{5}{\frac{1}{5}}$$

$$r = 25$$

The taxi travels 25 miles per hour.

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